CHAPTER 3

ARRAY THEORY

An antenna Array is a configuration of individual radiating elements that are arranged in space and can be used to produce a directional radiation pattern. Single-element antennas have radiation patterns that are broad and hence have a low directivity that is not suitable for long distance communications. A high directivity can be still be achieved with single-element antennas by increasing the electrical dimensions (in terms of wavelength) and hence the physical size of the antenna. Antenna arrays come in various geometrical configurations, the most common being; linear arrays (1D). Arrays usually employ identical antenna elements. The radiating pattern of the array depends on the configuration, the distance between the elements, the amplitude and phase excitation of the elements, and also the radiation pattern of individual elements.

3.1 Some Antenna parameter definitions

It is worthwhile to have a brief understanding of some of the antenna parameters before discussing antenna arrays in detail. Some of the parameters discussed in [2] are explained below.

3.1.1 Radiation Power density

Radiation Power density W_r gives a measure of the average power radiated by the antenna in a particular direction and is obtained by time-averaging the Poynting vector.

$$W_r(r,\theta,\phi) = \frac{1}{2}\mathrm{Re}\left[E \times H^*\right] = \frac{1}{2\eta}\left|\overline{E}(r,\theta,\phi)\right|^2 \quad \text{(Watts/m}^2\text{)} \tag{3.1}$$

where, E is the electric field intensity; H is the magnetic field intensity, and η is the intrinsic impedance

3.1.2 Radiation Intensity

Radiation intensity U in a given direction is the power radiated by the antenna per unit solid angle. It is given by the product of the radiation density and the square of the distance r.

$$U = r^2 W_r \quad \text{(Watts/unit solid angle)} \tag{3.2}$$

3.1.3 Total power radiated

The total power radiated P_{tot} by the antenna in all the directions is given by,

$$P_{tot} = \int_0^{2\pi}\int_0^{\pi} W_r(r,\theta,\phi) r^2 \sin(\theta) d\theta d\phi \tag{3.3}$$

$$= \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin(\theta) d\theta d\phi \quad \text{(Watts)} \tag{3.4}$$

3.1.4 Directivity

The Directive gain D_g, is the ratio of the radiation intensity in a given direction to the radiation intensity in all the directions. i.e.

$$D_g = \frac{4\pi U(\theta,\phi)}{P_{tot}}$$

$$= \frac{4\pi r^2 W_r(r,\theta,\phi)}{\int_0^{2\pi}\int_0^{\pi} W_r(r,\theta,\phi) r^2 \sin(\theta) d\theta d\phi} = \frac{4\pi U(\theta,\phi)}{\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin(\theta) d\theta d\phi} \tag{3.5}$$

The Directivity D_0 is the maximum value of the directive gain D_g for a given direction. i.e.

$$D_0 = \frac{4\pi U_{\max}(\theta,\phi)}{P_{tot}} \tag{3.6}$$

where $U_{\max}(\theta,\phi)$ is the maximum radiation intensity.

3.1.5 Radiation Pattern

The Radiation pattern of an antenna can be defined as the variation in field intensity as a function of position or angle. Let us consider an anisotropic radiator, which has stronger radiation in one direction than in another. The radiation pattern of an anisotropic radiator shown below in figure 3.1 consists of several lobes. One of the lobes has the strongest radiation intensity compared to other lobes. It is referred to as the *Major lobe*. All the other lobes with weaker intensity are called *Minor Lobes*. The width of the main beam is quantified by the *Half Power Beamwidth (HPBW)*, which is the angular separation of the beam between half-power points.

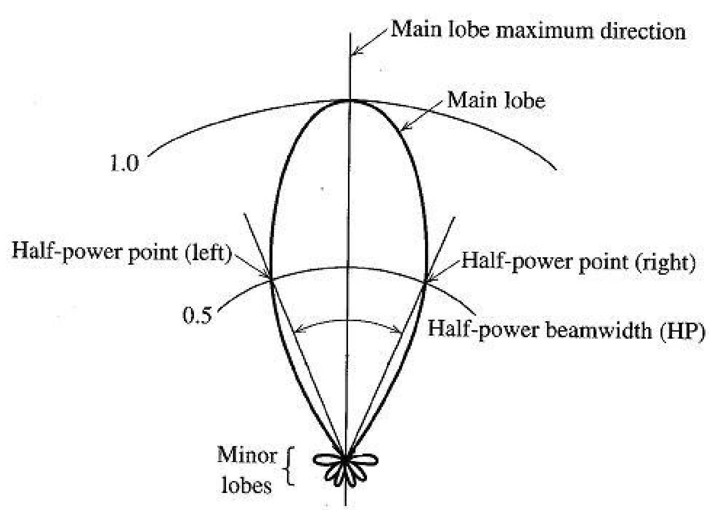

Figure 3.1 Radiation Pattern

3.2 Linear Array Analysis

When an antenna array has elements arranged in a straight line it is known as a linear array [2]. Let us consider a linear array with two elements shown in figure 3.2. The elements are placed on either sides of the origin at a distance $d/2$ from it.

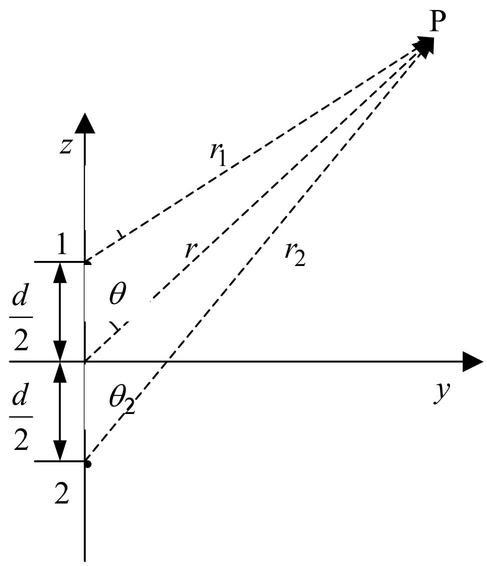

Figure 3.2 A two-element linear array

The electric field radiated by these two elements in the far field region at point *P* is of the following form.

Electric field at *P* due to element *1*:

$$\overline{E}_1 = w_1 f_1(\theta_1, \phi_1) \frac{e^{-j\left(kr_1 - \frac{\beta}{2}\right)}}{r_1} \tag{3.7}$$

Electric field at *P* due to element *2*:

$$\overline{E}_2 = w_2 f_2(\theta_2, \phi_2) \frac{e^{-j\left(kr_2 + \frac{\beta}{2}\right)}}{r_2} \tag{3.8}$$

Where:

w_1, w_2 are the weights;

f_1, f_2 are the normalized field patterns for each antenna element;

r_1, r_2 are the distances of element *1* and element *2* from the observation point P;

β is the phase difference between the feed of the two array elements;

To make the far field approximation the above figure can be re-drawn as shown below in figure 3.3. The point P is in the far field region.

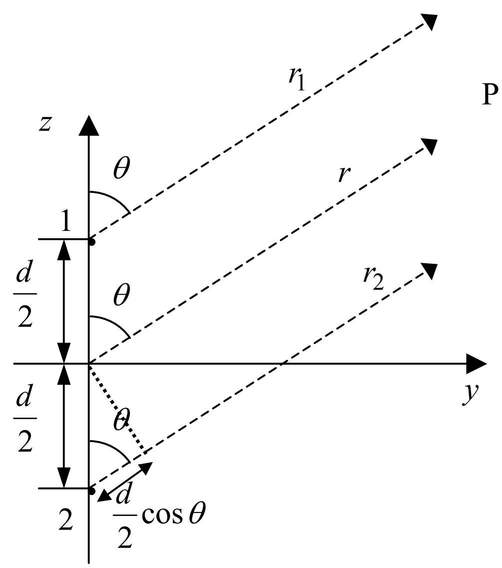

Figure 3.3 Far-field geometry of a two-element linear array

Following approximations can be drawn from the above diagram:

$\theta_1 \cong \theta_2 \cong \theta$;

$r_1 = r_2 = r\}$ For amplitude variations

$$\left.\begin{aligned} r_1 &\cong r - \frac{d}{2}\cos\theta \\ r_2 &\cong r + \frac{d}{2}\cos\theta \end{aligned}\right\} \text{For phase variations} \qquad (3.8)$$

Since the array elements are identical we can assume the following:

$$F_1(\theta_1,\phi_1)=F_2(\theta_2,\phi_2)=F(\theta,\phi) \tag{3.9}$$

The total field $\overline{E}$ at point P is the vector sum of the fields radiated by the individual elements and can we illustrated as follows:

$$\overline{E}=\overline{E}_1+\overline{E}_2$$

$$\overline{E}=w_1 f(\theta,\phi)\frac{e^{-j\left(k(r-\frac{d}{2}\cos\theta)-\frac{\beta}{2}\right)}}{r}+w_2 f(\theta,\phi)\frac{e^{-j\left(k(r+\frac{d}{2}\cos\theta)+\frac{\beta}{2}\right)}}{r} \tag{3.10}$$

$$\overline{E}=\frac{e^{-jkr}}{r}f(\theta,\phi)\underbrace{\left[w_1 e^{j\left(k\frac{d}{2}\cos\theta+\frac{\beta}{2}\right)}+w_2 e^{-j\left(k\frac{d}{2}\cos\theta+\frac{\beta}{2}\right)}\right]}_{AF} \tag{3.11}$$

For uniform weighting,

$$w_1=w_2=w \tag{3.12}$$

$$\vec{E}=w\frac{e^{-jkr}}{r}f(\theta,\phi)\times\underbrace{2\cos\left(\frac{kd\cos\theta+\beta}{2}\right)}_{AF} \tag{3.13}$$

The above relation as often referred to as pattern multiplication which indicates that the total field of the array is equal to the product of the field due to the single element located at the origin and a factor called *array factor, AF*. i.e.

(total) = [E(single element at reference point)] × [array factor] (3.14)

Note: The pattern multiplication rule only applies for an array consisting of identical elements.

The normalized array factor for the above two-element array can be written as follows:

$$AF_n = \cos\left(\frac{kd\cos\theta + \beta}{2}\right) \tag{3.15}$$

Therefore from the above discussion it is evident that the *AF* depends on:

1. The number of elements
2. The geometrical arrangement
3. The relative excitation magnitudes
4. The relative phases between elements

3.3 Uniform Linear Array

Based on the simple illustration of a two-element linear array let us extend the analysis to a N-element uniform linear array [2]. A uniform array consists of equispaced elements, which are fed with current of equal magnitude (i.e. with uniform weighting) and can have progressive phase-shift along the array.

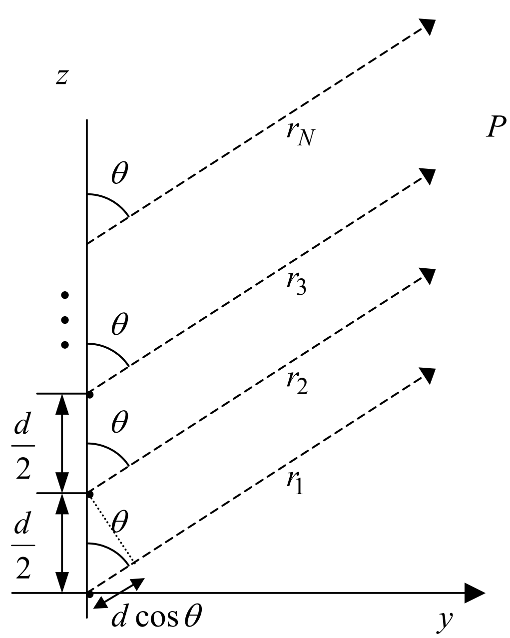

Figure 3.4 Far-field geometry of N-element array of isotropic elements along z-axis

The uniform linear array shown in the figure 3.4 consists of N elements equally spaced at distance d apart with identical amplitude excitation and has a progressive phase difference of β between the successive elements. Let us assume that the individual radiating elements are point sources with the first element of the array at the origin. The phase of the wave arriving at the origin is set to zero. Again point P is assumed to be in the far field region.

The AF of an N-element linear array of isotropic sources is:

$$AF = 1 + e^{+j(kd\cos\theta+\beta)} + e^{+j2(kd\cos\theta+\beta)} \cdots + + e^{+j(N-1)(kd\cos\theta+\beta)} \tag{3.16}$$

The above equation can be re-written as:

$$AF = \sum_{n=1}^{N} e^{+j(n-1)(kd\cos\theta+\beta)} \tag{3.17}$$

$$AF = \sum_{n=1}^{N} e^{+j(n-1)\psi} \tag{3.18}$$

where $\psi = kd\cos\theta + \beta$. Therefore by varying β the array factor of the array can be controlled. The above AF relation can be expressed in a closed form, which is more convenient for pattern analysis,

$$AF \cdot e^{j\psi} = \sum_{n=1}^{N} e^{jn\psi} \tag{3.19}$$

$$AF \cdot e^{j\psi} - AF = e^{j\psi} - 1 \tag{3.20}$$

$$AF = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = \frac{e^{j\frac{N}{2}\psi}\left(e^{j\frac{N}{2}\psi} - e^{-j\frac{N}{2}\psi}\right)}{e^{j\frac{\psi}{2}}\left(e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}\right)} \tag{3.21}$$

$$AF = e^{j\left(\frac{N-1}{2}\right)\psi} \cdot \frac{\sin\left[\frac{N}{2}\psi\right]}{\sin\left[\frac{\psi}{2}\right]} \tag{3.22}$$

In the above analysis the term $e^{j\left(\frac{N-1}{2}\right)\psi}$ is equal to one if the origin coincides with the center of the array. Therefore neglecting that term the array factor can be re-written as:

$$AF = \frac{\sin\left[\frac{N}{2}\psi\right]}{\sin\left[\frac{\psi}{2}\right]} \tag{3.23}$$

For small values of ψ the above equation can be reduced to:

$$AF = \frac{\sin\left[\frac{N}{2}\psi\right]}{\frac{\psi}{2}} \tag{3.24}$$

The maximum value of the array factors is N. Therefore their normalized form can be written as follows:

$$\boxed{AF_n = \frac{1}{N}\left[\frac{\sin\left[\frac{N}{2}\psi\right]}{\sin\left[\frac{\psi}{2}\right]}\right]} \tag{3.25}$$

or

$$\boxed{AF_n = \frac{1}{N}\left[\frac{\sin\left[\frac{N}{2}\psi\right]}{\frac{\psi}{2}}\right]}\text{, for small values of } \psi \tag{3.26}$$

3.3.1 Nulls and Maxima of the Array Factor

In order to find the nulls of the AF, the above AF equation 3.26 is set to zero. The analysis to find the angles θ_n at which the nulls occur is as follows:

$$\sin\left[\frac{N}{2}\psi\right] = 0 \Rightarrow \frac{N}{2}\psi = \pm n\pi \Rightarrow \frac{N}{2}(kd\cos\theta_n + \beta) = \pm n\pi \tag{3.27}$$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\right)\right], \qquad n = 1,2,3....(n \neq N,2N,3N,...) \tag{3.28}$$

There are no existing nulls when $n = N,2N,3N,...$, as the argument of the arccosine exceeds unity.

The angles θ_m at which the maxima occurs can be obtained when

$$\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}(-\beta \pm 2m\pi)\right] \tag{3.29}$$

If $\frac{d}{\lambda}$ is chosen to be sufficiently small the AF in equation (3.26) has only one maximum and it occurs when *m=0* in equation 3.29. i.e.,

$$\theta_m = \cos^{-1}\left[\frac{\lambda\beta}{2\pi d}\right] \tag{3.30}$$

For $m = 1,2...$ the argument of the arccosine function in equation (3.29) becomes greater than unity.

3.3.2 Half Power Beamwidth *(HPBW)(for the major lobe)*

In order to compute the *HPBW* in addition to the angle of first maximum θ_m, the half-power point θ_h is also required. The half-power point θ_h can be calculated by setting the value of AF_n in equation (3.26) to 0.707. i.e.

$$\frac{N}{2}\psi = \frac{N}{2}(kd\cos\theta_h + \beta) = \pm 1.391 \qquad (3.31)$$

$$\theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right] \qquad (3.32)$$

Now the *HPBW* can be calculated as:

$$HPBW = 2\left|\theta_m - \theta_h\right| \qquad (3.33)$$

Note: The *HPBW* equation provided here is only for non-steered arrays.

3.3.3 Broadside Array

An array is referred to as a broadside array when it has a maximum radiation in the direction perpendicular to that of axis of the array i.e. when $\theta = 90$ as shown in figure 3.5. From the equation (3.29) the maximum of the array factor would occur when

$$\psi = kd\cos\theta + \beta = 0 \quad (\text{for } m = 0) \qquad (3.34)$$

If $\theta = 90$ then,

$\Rightarrow \beta = 0$

Therefore a uniform linear array will have maximum radiation in the broadside direction when all the array elements will have same phase excitation.

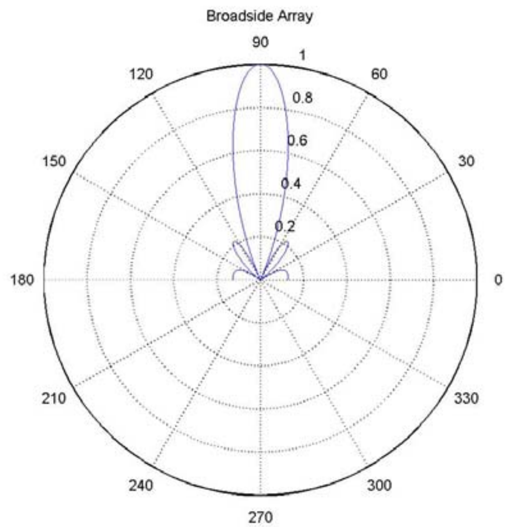

Figure 3.5 Array factor of a 5-element uniform amplitude broadside array

3.3.4 End-fire Array

An array is referred to as an end-fire array when it has a maximum radiation in the direction along the axis of the array i.e. when $\theta = 0^\circ$ (as shown in figure 3.6) or $\theta = 180^\circ$.

When $\theta = 0^\circ$,

$$\psi = kd\cos\theta + \beta = kd + \beta = 0$$

$$\Rightarrow \beta = -kd \tag{3.35}$$

When $\theta = 180^\circ$,

$$\psi = kd\cos\theta + \beta = -kd + \beta = 0$$

$$\Rightarrow \beta = kd \tag{3.36}$$

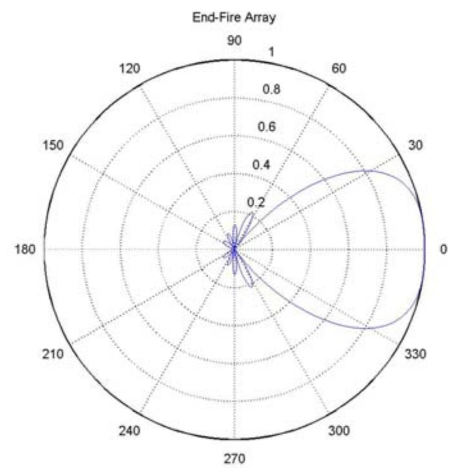

Figure 3.6 Array factor of a 5-element uniform amplitude end-fire array

Let us now illustrate the dependence of the uniform linear array factor on various parameters including the number of elements N and element spacing d as a function of wavelength λ.

The array factor plots shown below indicates that beamwidth inversely proportional to the spacing d between the elements for same number of elements. The array factor plot in figure 3.7 shows that beam width is smaller in the first case when $d = \lambda/2$ compared to the beamwidth when $d = \lambda/4$.

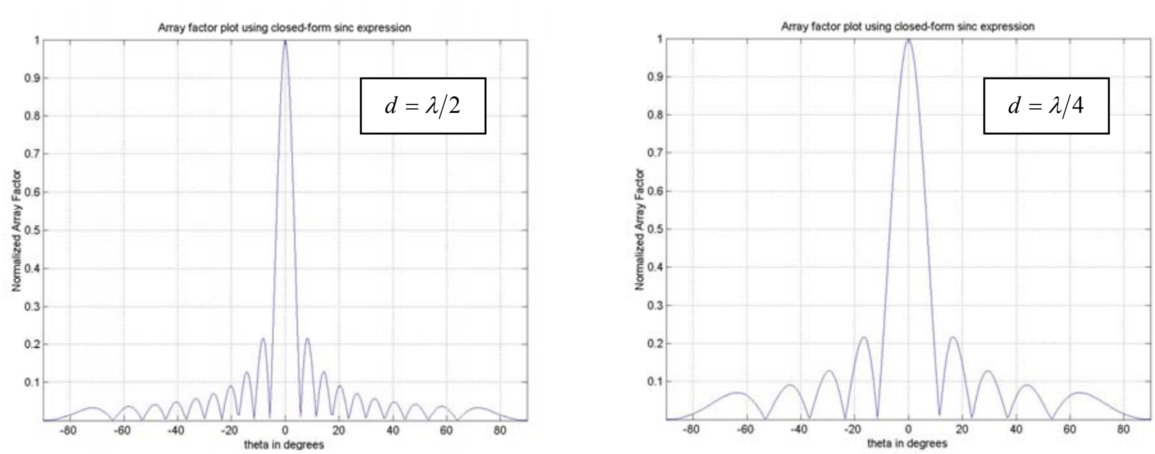

Figure 3.7 Linear plots of the Array factor plots for $d = \lambda/4$ and $d = \lambda/2$ when $N = 20$

The following array factor plots in figure 3.8 shows that the beamwidth is not only dependent on the element spacing d but also on the number of elements N. It is quite evident from the plots that the beamwidth increases as the number of elements in the array increases. Please note that the element spacing d is kept constant in both the cases.

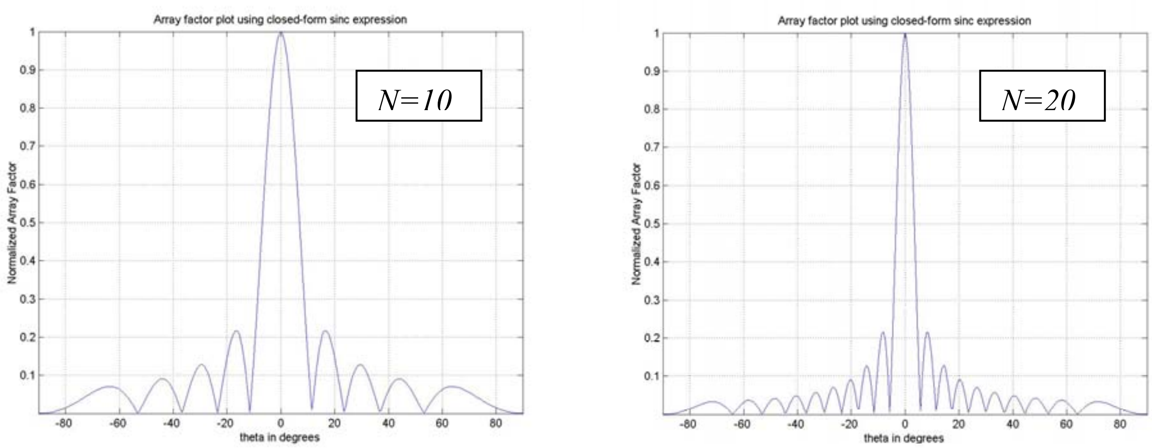

Figure 3.8 Linear plots of the Array factor plots for $N = 10$ and $N = 20$ when $d = \lambda/2$

From the plots shown in figure 3.9 it is evident that beamwidth remains the same if number of elements N is increased and element spacing d is decreased by the same amount. Here number of elements N is increased from 10 to 20; at the same time element spacing is decreased from $d = \lambda/2$ to $d = \lambda/4$ and there is no change in beamwidth.

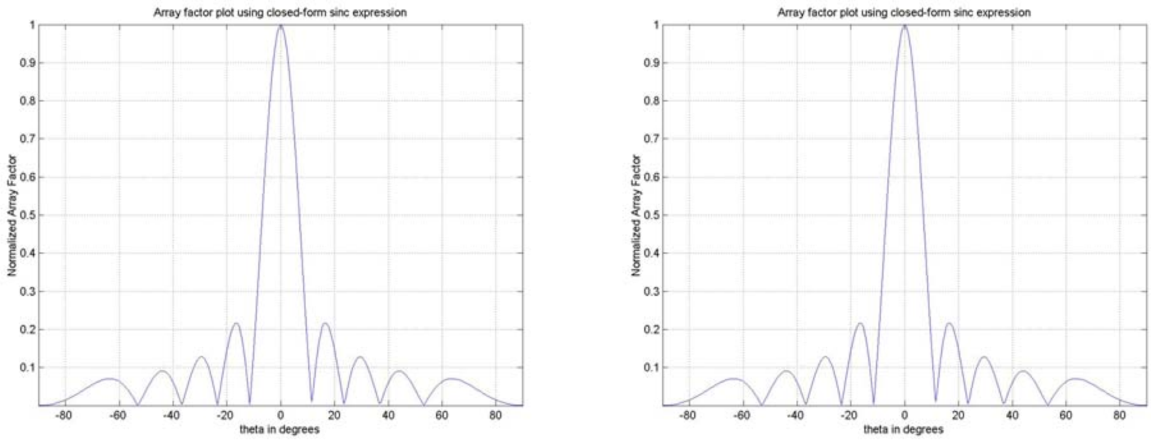

Figure 3.9 Linear plots of the Array factor plots for $N = 10$, $d = \lambda/2$ and $N = 20$, $d = \lambda/4$

3.4 Phased (scanning) Arrays

From the above discussion on broadside array and end-fire array it is quite obvious that the direction of the radiation for the main beam (m=0) depends on the phase difference β between the elements of the array. Therefore it is possible to continuously steer the main beam in any direction by varying the progressive phase β between the elements. This type of array where the main beam is steered to the desired direction is referred to as a phased array or a phase-scanned array.

The array factor for an N-element linear array with uniform spacing is given by,

$$AF(\theta) = \sum_{n=0}^{N-1} w_n e^{jknd\cos(\theta)} \tag{3.37}$$

The phase β is adjusted by including a phase factor $e^{-jkd\cos\theta_0}$ in the weight associated with individual element.

If the steering angle desired is θ_0 the phase excitation β must be adjusted such that

When $\theta = \theta_0$,

$$\psi = kd\cos\theta + \beta = kd\cos\theta_0 + \beta = 0$$

$$\Rightarrow \beta = -kd\cos\theta_0 \tag{3.38}$$

The normalized AF of a beam steered array is given as

$$AF_n = \frac{\sin\left(N\pi\dfrac{d}{\lambda}\left(\cos\theta - \cos\theta_0\right)\right)}{N\pi\dfrac{d}{\lambda}\left(\cos\theta - \cos\theta_0\right)}, \text{ for small values of } \psi \tag{3.39}$$

The figure 3.10 shows a linear and a polar plot, which illustrates the above discussion on how the beam is steered to a desired angle θ_0 (in this case $\theta_0 = 40^0$) by using normalized AF relation. The following case is for a 20-element array and the elements are spaced $\frac{\lambda}{2}$ apart.

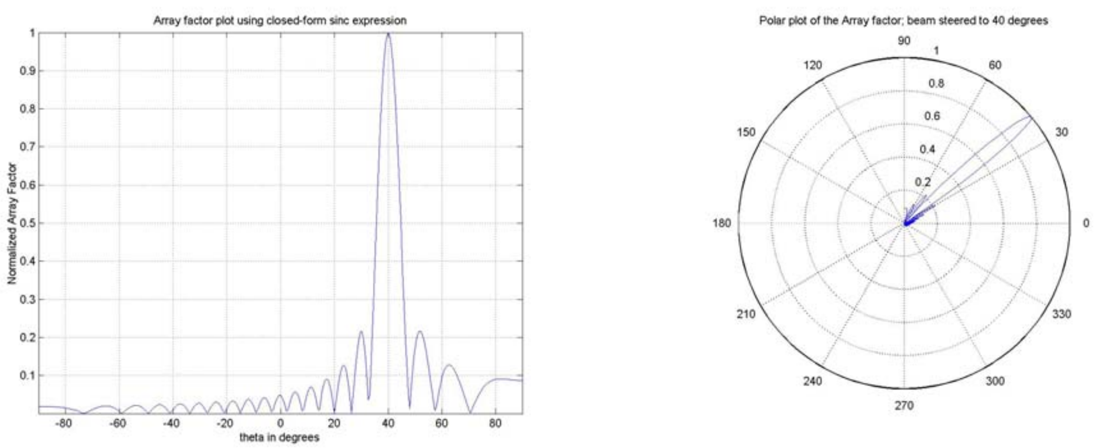

Figure 3.10 Linear and polar plots of the array factor when beam is steered to 40 degrees